Comprehensive set of analytical solutions for two-dimensional advective-dispersive transport involving flexible boundary inputs, initial distributions and zero-order productions

You-Lin Tu¹  Jui-Shang Chen¹  Keng-Hsin Lai¹
¹Graduate Institute of Applied Geology, National Central University, Zhongli City, Taoyuan County 32001

Abstract
A comprehensive set of analytical solutions for the two-dimensional advective-dispersion equation in a finite domain involving a wide variety of boundary inputs, initial distributions, and zero-order productions are presented in this study. First, the generalized analytical solutions are obtained by successively applying different integral transforms corresponding to the governing equations and its associated initial and boundary conditions. Based on the generalized analytical formulation, a comprehensive set of special-case solutions for some time-dependent boundary distributions and zero-order productions were described by Dirac delta, constant, Heaviside, exponentially-decaying, or periodically sinusoidal functions as well as some position-dependent initial conditions and zero-order productions produced by Dirac delta, constant, Heaviside, or exponentially-decaying functions are derived. Several sample applications, which are rarely noted in the literature, are given based on the comprehensive set of special-case solutions. The solution strategy presented in this study can be applied to more complicated scenarios of solute transport subjected to sequential decay chain reactions.

Mathematical model

Governing equation
\[ \frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + \frac{\partial C}{\partial t} + S_p(x,t) \delta(y) \]

Initial condition
\[ C(x,y,t=0) = S_p(x) \delta(y) \]

Boundary conditions
- \[ \frac{\partial C}{\partial x}(x=0,y,t) = 0 \]
- \[ \frac{\partial C}{\partial x}(x=L_x,y,t) = 0 \]
- \[ \frac{\partial C}{\partial y}(x,y=0,t) = 0 \]
- \[ \frac{\partial C}{\partial y}(x,y=W_y,t) = 0 \]

Solution
\[ C(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F(n,m) \theta(x) \sin(m \pi y / W_y) e^{-\beta^2 \pi^2 D_y t / W_y^2} \]

More results and animes

Table 1. The comparison of the breakthrough curves obtained from periodically sinusoidal functions and numerical solutions for various dispersion coefficients and a periodic constant value.

Table 2. The comparison of the breakthrough curves at \( x = 0 \) m and \( y = 5 \) m obtained from periodically sinusoidal functions and numerical solutions for various dispersion coefficients and a periodic constant value.

Fig. 1. The composite of the concentration contours obtained from the proposed analytical solutions and those from the corresponding numerical solution at \( t = 5 \) days.

Fig. 2. Temporal evolution of the breakthrough curves at \( y = 5 \) m obtained from two analytical solutions and numerical solutions of Heaviside function: \( C(x,y,t) = \theta(x) \delta(y) \exp(-\alpha \pi^2 D_y t / W_y^2) \).

Fig. 3. Simulates two sources distribution from source.