Exact analytical solutions to the advection-dispersion equation in a radially divergent flow field

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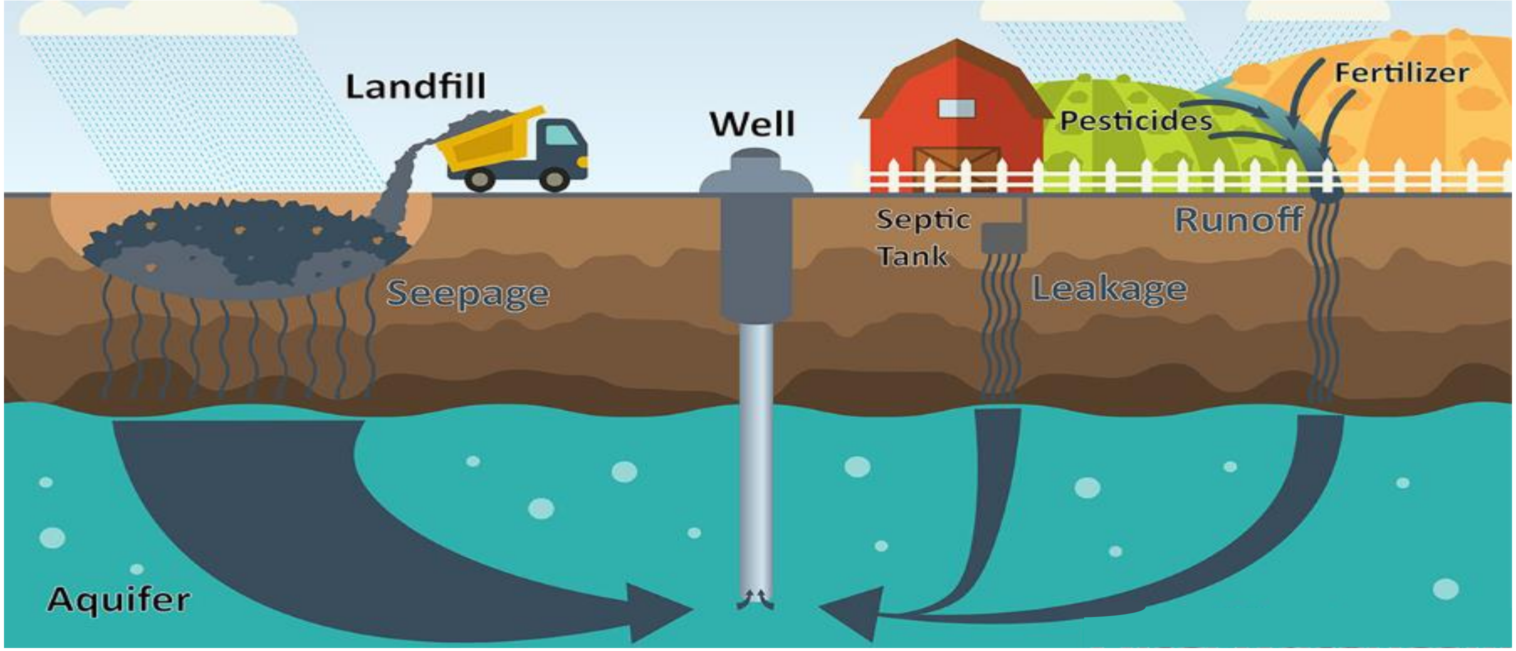
Date:2023/3/17

Outline

Groundwater contamination

- Problem with groundwater contamination have grown increasingly severe worldwide over the past few decades.
- Contaminant in the aquifer will be affected by physical, chemical effects to change their concentration or transform into other contaminant.
- It is important to understand about transport of contaminant and fate in groundwater system.



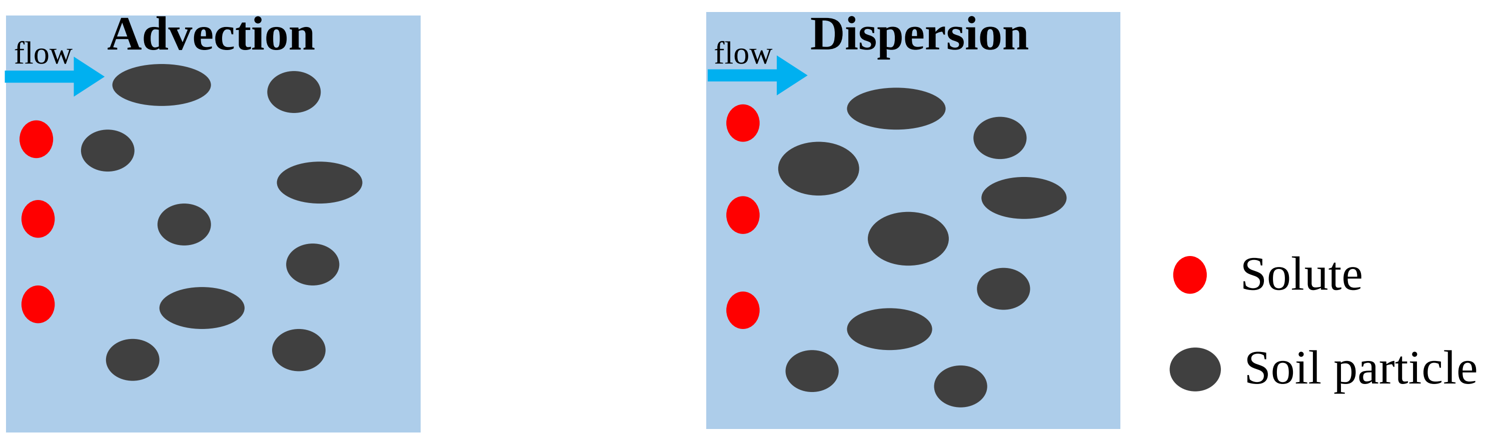
Mathematical model

- Advection-dispersion equation (ADE) is commonly used to predict the fate and transport of contaminants in groundwater.

- Mathematical models (numerical methods or analytical methods) based on the ADE are general tools to estimate the reactive migration in geological formations.

$$D\frac{\partial^2 C(x,t)}{\partial x^2} - v\frac{\partial C(x,t)}{\partial x} = \mathrm{R}\frac{\partial C(x,t)}{\partial t}$$

Simple example for ADE



Analytical model

Analytical models are highly sought after as they provide greater insight into the governing transport processes. (Carr, 2021)

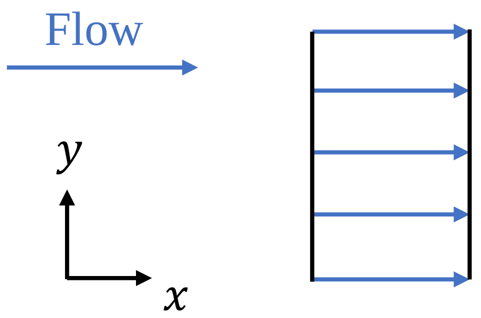
Advantages of analytical model

- High computational efficiency.
- Doesn’t require the use of small temporal and spatial discretization step sizes.
- Extrapolating results over large times or extensive spatial scales.
- Easy to code into a computer program.

Flow field

Uniform flow field

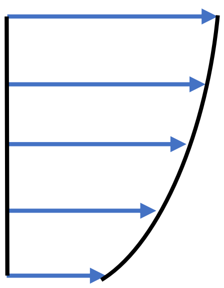
Flow velocity does not change



- To describe solute transport in geological formations, a number of analytical models have been derived for describing in a uniform flow field.

Non-uniform flow field

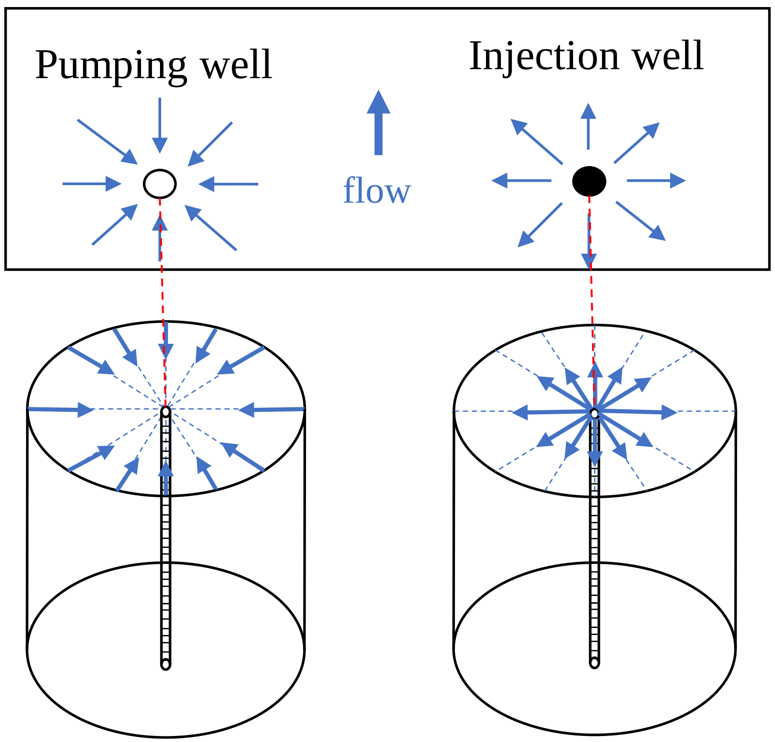
Flow velocity will vary with position



- Finding analytical solutions for solute transport in a non-uniform flow field is difficult and relative rare. (Lin., 2016)

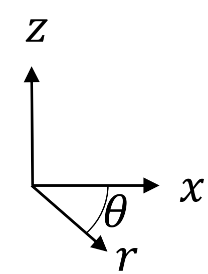
Radial flow field

Solute transport in a radial flow field created by an injection/pumping well can be viewed as a special case of solute transport in a non-uniform flow field. (Chen, 2016)



Convergent flow field   Divergent flow field

Cylindrical coordinate system



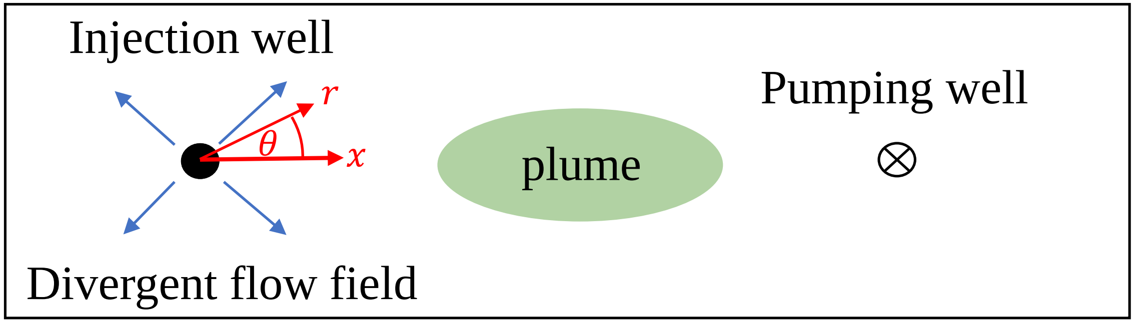
The average pore velocity in the radial direction is:

$$v(r) = \frac{Q}{2\pi b\phi}\frac{1}{r}$$

r : radial distance [L]

Cylindrical coordinate system

The ADE in cylindrical coordinates refers to the problem of analyzing the dispersive transport of a contaminant in the radial flow field generated.



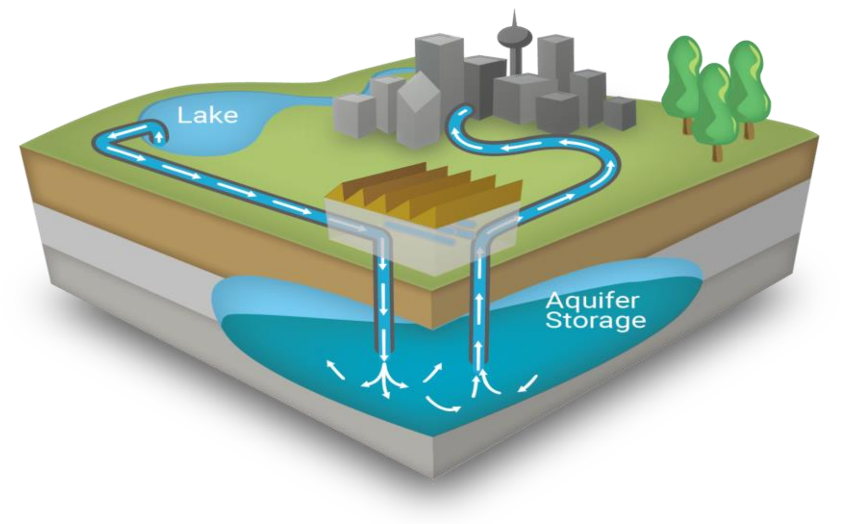
$$\frac{1}{r}\frac{\partial}{\partial r}[rD(r)\frac{\partial C\ (r,t)}{\partial r}] - v(r)\frac{\partial C\ (r,t)}{\partial r} = R\frac{\partial C\ (r,t)}{\partial t}$$

One-dimensional radial flow field for ADE

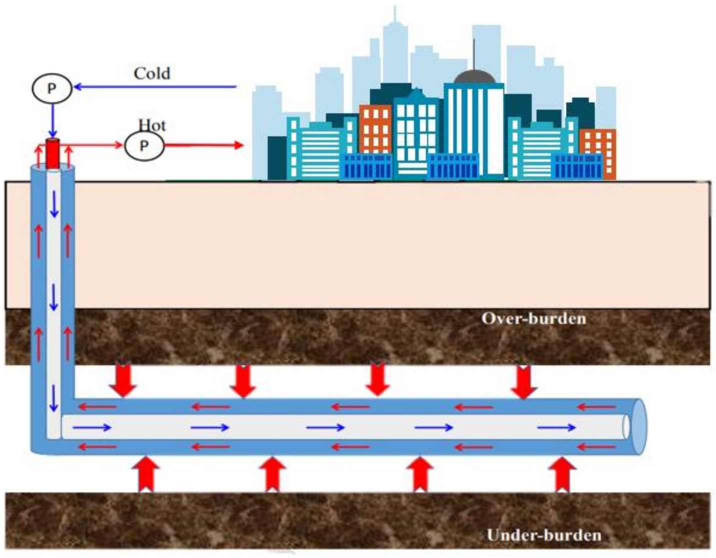
- These models have been widely used to describe solute transport processes, such as contaminant in porous media, heat transport in geothermal reservoirs.

Application of radial flow field model

Radial flow fields models have many practical applications, which involves aquifer storage and recovery (ASR), geothermal development, aquifer decontamination by pumping and tracer test.



ASR



Geothermal well (Sun, 2018)

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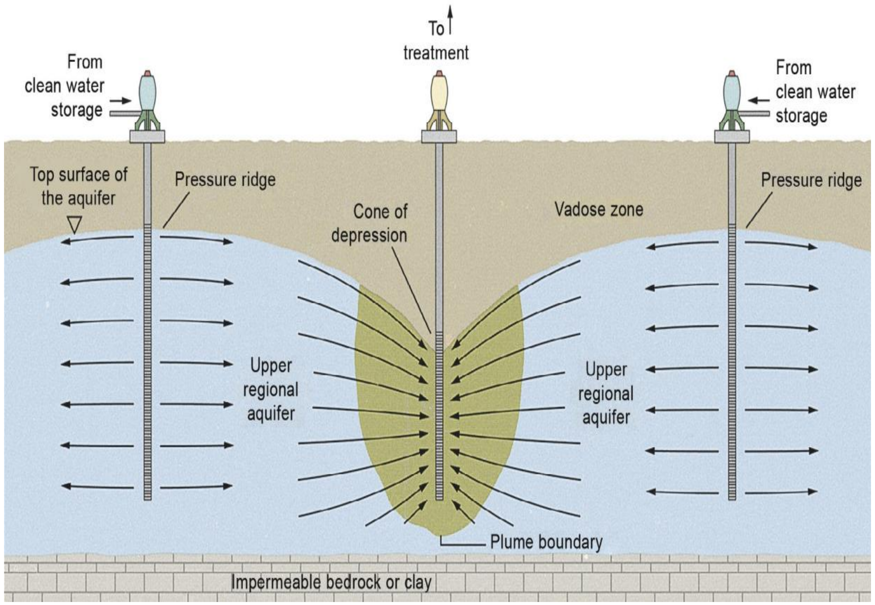
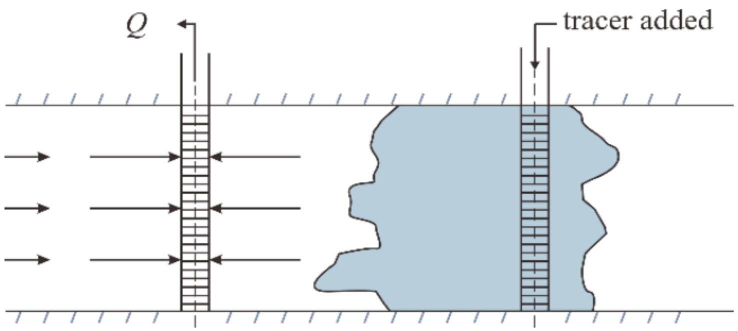


Aquifer decontamination by pumping

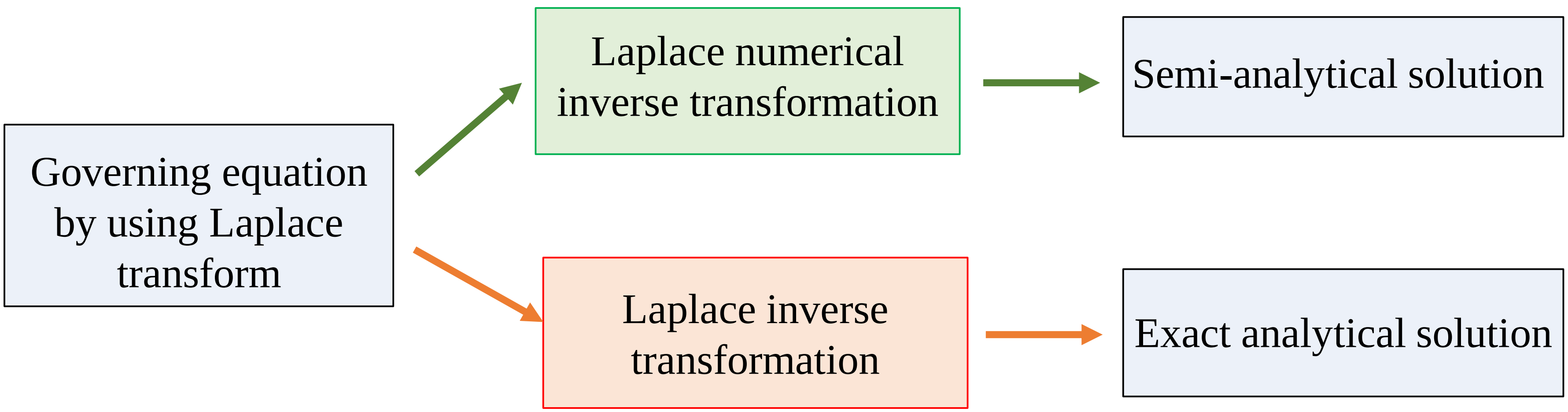


Tracer test

Literature Review

Semi-analytical solutions

- Many studies develop analytical solutions to the radial advection-diffusive solute transport problem, but most of the models are semi-analytical solutions.
(Ogata, 1958; Chen, 1985; Chen et al., 2002; Chen., 2010; Liu et al., 2013)



- Develop an exact analytical solution for the radial advection-dispersion transport is very difficult. (Wang and Zhan., 2015)

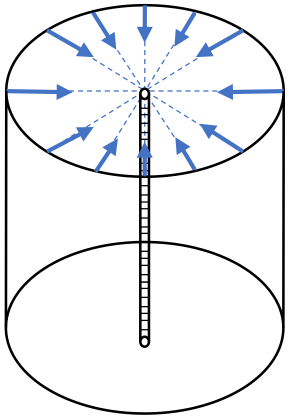
Literature Review

Chen et al., 2016

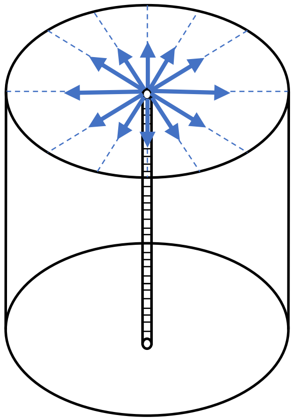
This research develops an exact analytical solution for the radial advection-dispersion transport.

- It adopts a relatively novel method makes it easier to perform mathematical operations than in the past.

However, this model can only simulate the transport phenomena caused by convergent flow field.



Convergent flow field



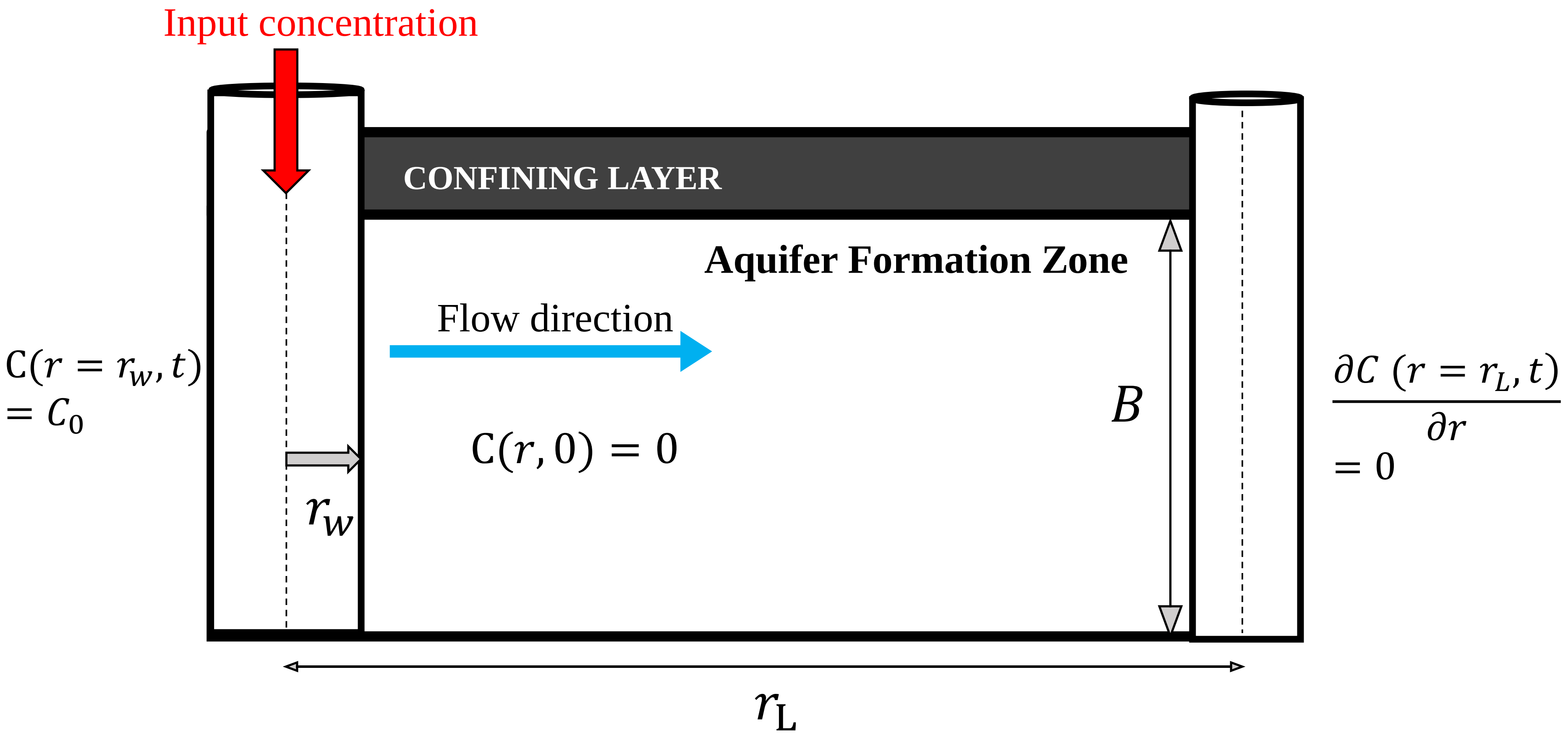
Divergent flow field

Objective

To develop an exact analytical solution for the advection-dispersion equation (ADE) in a radially divergent flow field. Using divergent flow tracer test to demonstrate the robustness of my model.

Methodology

Conceptual model



r_w : the radius of the injection $[L]$
r_L : distance between the injection and extraction wells $[L]$
B: aquifer thickness $[L]$
C_0: source concentration of the injection $[ML^{-3}]$

Governing equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\frac{\partial C\,(r,t)}{\partial r}\right] - v(r)\frac{\partial C\,(r,t)}{\partial r} = R\frac{\partial C\,(r,t)}{\partial t}, r_{\mathrm{w}} \le r \le r_L, t > 0$$

Dispersion term **Advection term** **sorption**

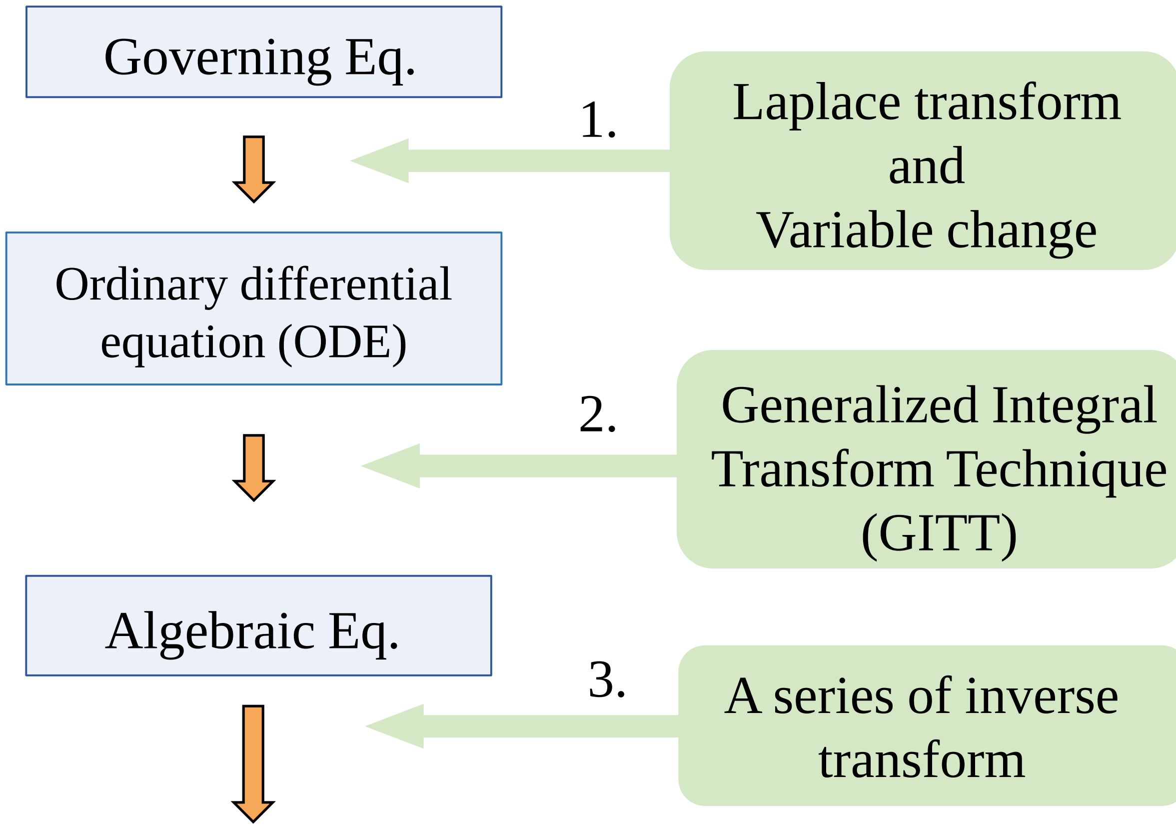
$$D(r) = \alpha v(r)$$

$$v(r) = \frac{Q}{2\pi b\phi r}$$

C : concentration $[ML^{-3}]$
t : time since injection [T]
r : radial distance [L]
D : dispersion coefficients [L^2T^{-1}]
v : groundwater velocity $[LT^{-1}]$

ϕ: effective porosity [-]
α: dispersivitie [L]
R: retardation coefficient [-]
Q: groundwater injection rate $[L^3T^{-1}]$

Analytical solution derivation



Exact analytical solution: $C_D(\rho,\tau) = 1 - \exp(\frac{Pe\rho}{2}) \sum_{m=1}^{\infty} \exp(\frac{-\beta_m^2}{R}\tau) \frac{\Phi(\beta_m) K(\rho,\beta_m)}{N(\beta_m)}$

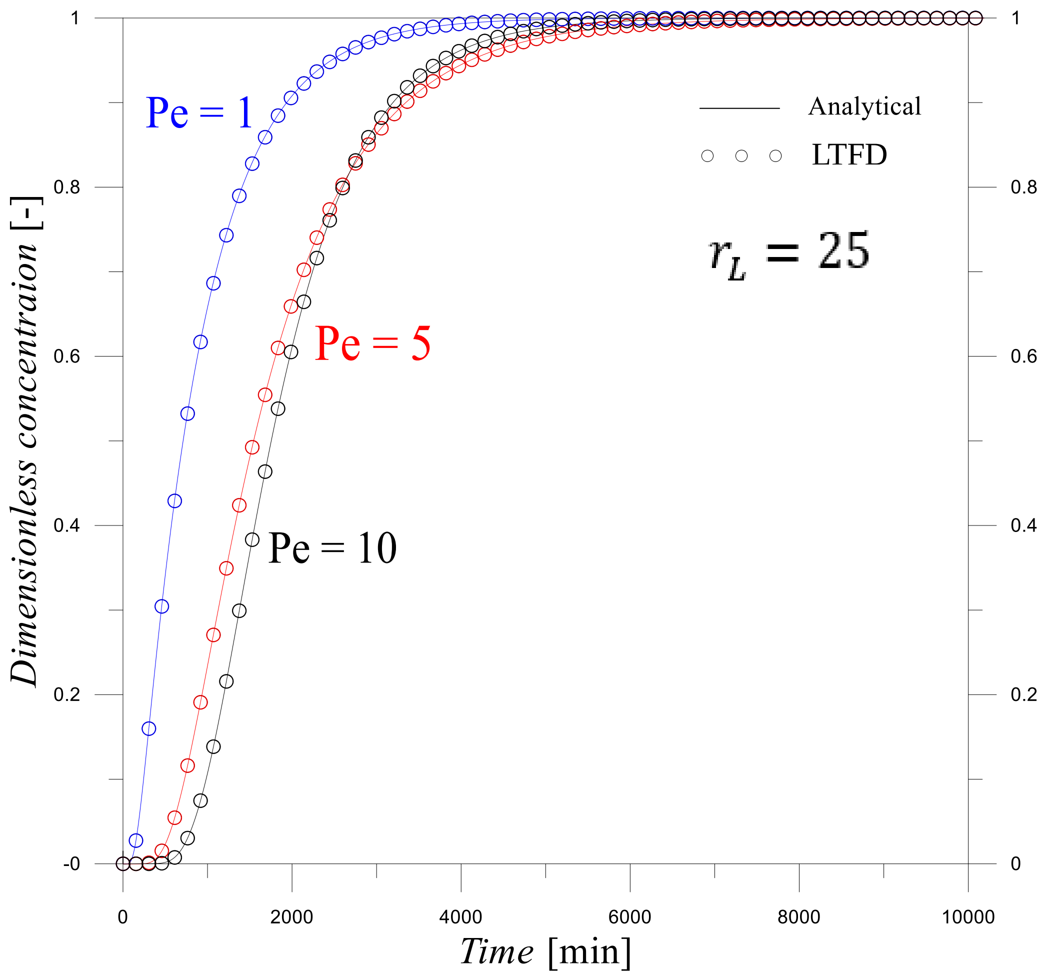
Results and Discussion

Parameters

Parameters	Values
Well radial , r_w [L]	0.1
Distance between the injection and extraction wells , r_L [L]	25
Aquifer thickness, b[L]	10
Initial concentration, $C_0[ML^{-3}]$	40
Radial dispersivity, α[L]	25/5/2.5
Effective porosity, ϕ[-]	0.2
Retardation coefficient, R[-]	1
Constant injection rate, Q[L^3T^{-1}]	2
Peclet number	**1/5/10**

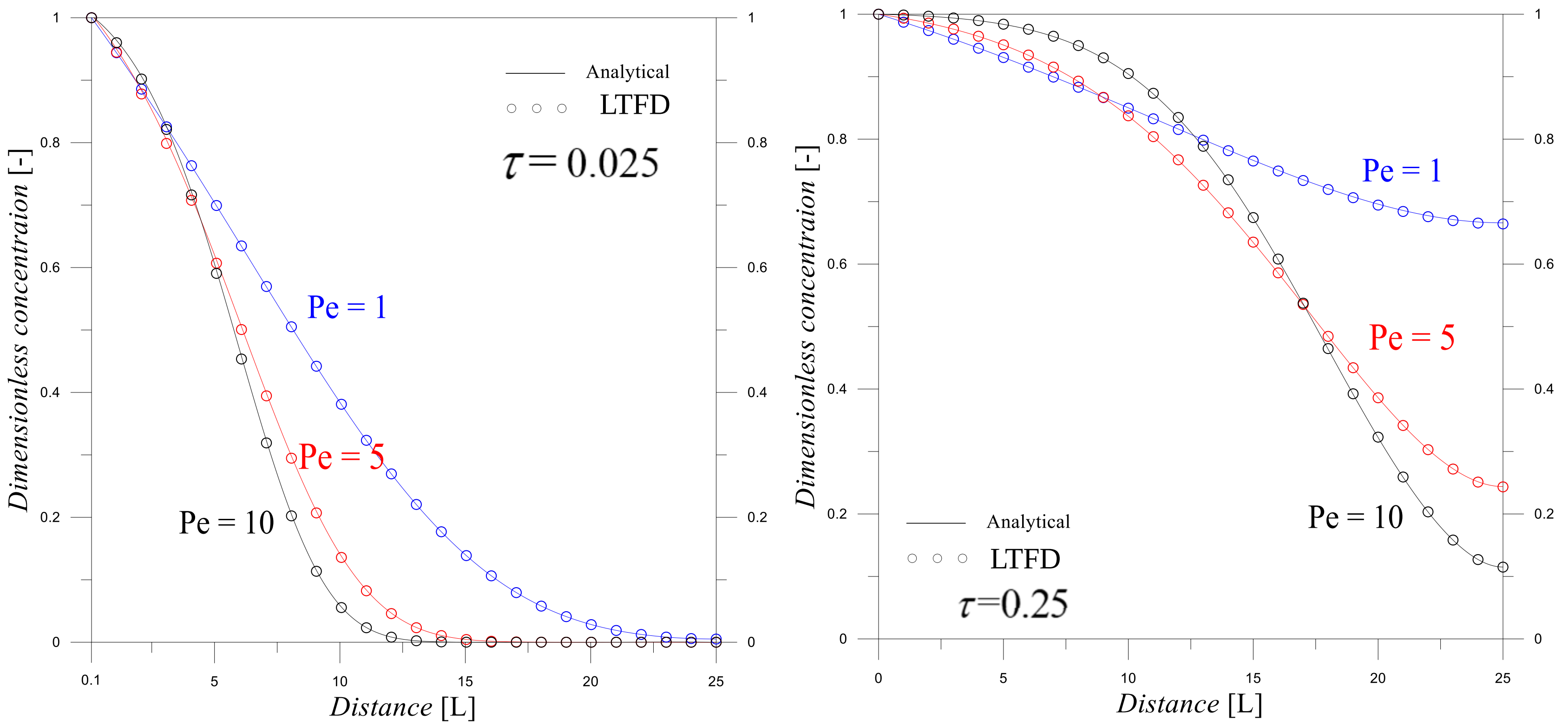
$$\text{Peclet number(Pe)} = \frac{v(r)r_L}{D(r)} = \frac{r_L}{\alpha}$$

Comparing with Laplace transform finite difference (LTFD)



Breakthrough Curves(BTCs)

Comparing with Laplace transform finite difference (LTFD)



Effect of the simulation time

Conclusions

Conclusions

- Develop an exact analytical solutions for advection-dispersion transport equation in a radial divergent flow field by using a very novel solution method.

- Compare the derived exact analytical solution with the LTFD, including BTCs and effect of the simulation time. The results show an excellent consistent.

Future work

- I will do the sensitivity analysis for sorption.

Thanks